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Zeta functions of digraphs

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Abstract

We define a zeta function of a digraph and an L-function of a symmetric digraph, and give determinant expressions of them. Furthermore, we give a decomposition formula for the zeta function of a g-cyclic Γ-cover of a symmetric digraph for any finite group Γ and $g \in \Gamma$.

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1. Introduction

Graphs and digraphs treated here are finite and simple. Let $G = (V(G), E(G))$ be a connected graph with vertex $V(G)$ and arc set $E(G)$, and D the symmetric digraph corresponding to G. Note that $E(G) = E(D)$. For $e = (u, v) \in E(G)$, let $o(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by $\bar{e}$. A *path P of length n* in D(or G) is a sequence $P = (v_0, v_1, \ldots, v_{n-1}, v_n)$ of $n + 1$ vertices and n arcs(or edges) such that consecutive vertices share an arc(or edge) (we do not require that all vertices are distinct). Also, P is called a *(v_0, v_n)-path*. We say that a path has a *backtracking* if a subsequence of the form $\ldots, x, y, x, \ldots$ appears. A (v, w)-path is called a *cycle* (or *closed path*) if $v = w$.

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We introduce an equivalence relation between cycles. Such two cycles $C_1 = (v_1, \ldots, v_m)$ and $C_2 = (w_1, \ldots, w_m)$ are called *equivalent* if $w_j = v_{j+k}$ for all j. Let $[C]$ be the equivalnce class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is said to be *reduced* if both C and C^2 have no backtracking. A cycle C is *prime* if $C \neq B^r$ for some other cycle B and $r \geqslant 2$.

The (*Ihara*) *zeta function* of a graph G is defined to be a formal power series of a variable u, by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} \left(1 - u^{|C|}\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C (see [13]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [8]. In [8], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada [15,16]. Hashimoto [7] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized the Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial.

Theorem 1 (Bass [2]). *Let G be a connected graph. Then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ is the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i \, (V(G) = \{v_i, \ldots, v_n\})$.

Stark and Terras [14] gave an elementary proof of Theorem 1, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass's Theorem were given by Foata and Zeilberger [5], Kotani and Sunada [10]. Mizuno and Sato [12] obtained a decomposition formula for the zeta function of a regular covering of a graph.

Foata and Zeilberger [5] gave a new proof of Bass's Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ a totally order in X, and X^* the free monoid generated by X. Then the totally order $<$ on X derive the lexicographic order $<$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r \geqslant 2$, and which is also minimal in the class of its cyclic rearrangements under $<$(see [9]). Let L denote the set of all Lyndon words in X.

Let $\mathbf{B}$ be a square matrix whose entries $b(x, x')(x, x' \in X)$ form a set of commuting variables. If $w = x_1 x_2 \cdots x_m$ is a word in X^*, define

$$\beta(w) = b(x_1, x_2)b(x_2, x_3) \ldots b(x_{m-1}, x_m)b(x_m, x_1).$$

Furthermore, let

$$\beta(L) = \prod_{l \in L} (1 - \beta(l)).$$

The following theorem played a central role in [5].

Theorem 2 (Foata and Zeilberger [5]). $\beta(L) = \det(\mathbf{I} - \mathbf{B})$.

Foata and Zeilberger[5] gave a short proof of Amitsur's identity [1] by using Theorem 2.

Theorem 3 (Amitsur [1]). *For square matrices* $\mathbf{A}_1, \ldots, \mathbf{A}_k$,

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \ldots, k\}$, *and* $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ *for* $l = i_1 \cdots i_p$.

In Section 2, we define a zeta function of a digraph, and give a determinant expression and an Euler product expression of it. In Section 3, we give a decomposition formula for the zeta function of a g-cyclic Γ-cover of a symmetric digraph for any finite group Γ and $g \in \Gamma$. In Section 4, we introduce an L-function of a symmetric digraph D, and express it by using the characteristic polynomial of some matrix. Furthermore, we show that the zeta function of D is a product of L-functions of D.

For a general theory of the representation of groups, the reader is referred to [3].

2. Zeta functions of digraphs

Let D be a connected digraph, and N_m the number of all cycles with length m in D. Then, the *zeta function* of a digraph D is defined to be a formal power series of a variable u, by

$$\mathbf{Z}_D(u) = \exp\left(\sum_{m \geqslant 1} \frac{N_m}{m} u^m\right).$$

Let D have n vertices $v_1, \ldots, v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(D) = (a_{ij})$ of D is the square matrix of order n such that $a_{ij} = 1$ if there exists an arc starting at the vertex v_i and terminating at the vertex v_j, and $a_{ij} = 0$ otherwise.

Theorem 4. *Let D be a connected digraph. Then the reciprocal of the zeta function of D is given by*

$$\mathbf{Z}_D(u)^{-1} = \det(\mathbf{I} - \mathbf{A}(D)u) = \prod_{[C]} \left(1 - u^{|C|}\right),$$

where $[C]$ *runs over all equivalence classes of prime cycles of D.*

Proof. Let $V(D)=\{v_1,\ldots,v_n\}$ and $v_1<v_2<\cdots<v_n$ a totally order of $V(D)$. We consider the free monoid $V(D)^*$ generated by $V(D)$, and the lexicographic order on $V(D)^*$ derived from $<$. If a cycle C is prime, then there exists a unique cycle in $[C]$ which is a Lyndon word in $V(D)$.

For $w \in V(D)^*$, let

$$\beta(w)=\begin{cases} u^{|w|} & \text{if } w \text{ is a prime cycle,}\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$\beta(L)=\prod_{l\in L}(1-\beta(l))=\prod_{[C]}\left(1-u^{|C|}\right),$$

where $[C]$ runs over all equivalence classes of prime cycles of D. Furthermore, we define variables $b(x,x')(x,x'\in V(D))$ as follows:

$$b(x,x')=\begin{cases} u & \text{if } (x,x')\in E(D),\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 2 implies that

$$\prod_{[C]}\left(1-u^{|C|}\right)=\det(\mathbf{I}-\mathbf{B})=\det(\mathbf{I}-u\mathbf{A}(D)).$$

Since $|[C]| = |C|$ and $N_m=\mathrm{tr}(\mathbf{A}(D)^m)$(see [4]), we have

$$\begin{aligned}\prod_{[C]}\left(1-u^{|C|}\right)^{-1} &= \exp\left(-\sum_{[C]}\log(1-u^{|C|})\right)\\ &= \exp\left(\sum_{[C]}\sum_{m\geqslant 1}\frac{1}{m}u^{|C|m}\right)\\ &= \exp\left(\sum_{m\geqslant 1}\sum_{C}\frac{1}{|C|m}u^{|C|m}\right)\\ &= \exp\left(\sum_{m\geqslant 1}\frac{N_m}{m}u^m\right).\end{aligned}$$

Therefore the result follows. □

The formula $\det(\mathbf{I}-\mathbf{A}(D)u)=\prod_{[C]}(1-u^{|C|})$ is also a specialization of Theorem 3.

Recently, Kotani and Sunada [10] treated zeta functions of strongly connected digraphs. In [10], they stated a connection between zeta functions of graphs and that of strongly connected digraphs, and gave a new proof of Bass's Theorem by using the connection. Let $G=(V,E)$ be a connected non-circuit graph. Then the *oriented line graph* $L(\vec{G})=(V_L,E_L)$ of G is defined as follows:

$$V_L = E;\ E_L = \{(e_1, e_2) \in E \times E \mid \bar{e}_1 \neq e_2, t(e_1) = o(e_2)\}.$$

There exist no reduced cycles in the oriented line graph. Thus, there is a one-to-one correspondence between prime cycles in $L(\vec{G})$ and prime, reduced cycles in G, and so $\mathbf{Z}_G(u) = \mathbf{Z}_{L(\vec{G})}(u)$. Furthermore, this is obtained from Theorem II.1.5 of Bass [2].

3. Zeta functions of cyclic Γ-covers

Let D be a symmetric digraph and Γ a finite group. A function $\alpha : E(D) \longrightarrow \Gamma$ is called *alternating* if $\alpha(y, x) = \alpha(x, y)^{-1}$ for each $(x, y) \in E(D)$. For $g \in \Gamma$, a *g-cyclic Γ-cover* $D_g(\alpha)$ of D is the digraph defined as follows (see [11]):

$$V(D_g(\alpha)) = V(D) \times \Gamma,$$

and

$$((v, h), (w, k)) \in E(D_g(\alpha)) \quad \text{if and only if } (v, w) \in E(D) \text{ and } k^{-1}h\alpha(v, w) = g.$$

The *natural projection* $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic Γ-cover* of D if D' is a g-cyclic Γ-cover of D for some $g \in \Gamma$.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : E(G) \longrightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph* G^α of the ordinary voltage graph (G, α) is defined as follows (see [6]):

$$V(G^\alpha) = V(G) \times \Gamma$$

and

$$((v, h), (w, k)) \in E(G^\alpha) \quad \text{if and only if } (v, w) \in E(G) \text{ and } k = h\alpha(v, w).$$

Similarly to the case of a cyclic Γ-cover of a symmetric digraph, the *natural projection* $\pi : G^\alpha \longrightarrow G$ is defined. The graph G^α is called a *derived graph covering* of G with voltages in Γ or an *Γ-covering* of G. The pair (D, α) of D and α can be considered as the ordinary voltage graph $(\tilde{D}, \alpha)$ of the underlying graph $\tilde{D}$ of D. Thus the 1-cyclic Γ-cover $D_1(\alpha)$ corresponds to the Γ-covering $\tilde{D}^\alpha$, where 1 is the unit of Γ.

Now, we give an example. Let D be the symmetric digraph of Fig. 1 and $\Gamma = Z_3 = \{0, 1, -1\}$ (the additive group). Furthermore, let $\alpha : E(D) \longrightarrow Z_3$ be the alternating function such that $\alpha(1, 2) = 0$, $\alpha(2, 3) = 1$ and $\alpha(3, 1) = -1$. Then the 1-cyclic Z_3-cover $D_1(\alpha)$ is shown in Fig. 2.

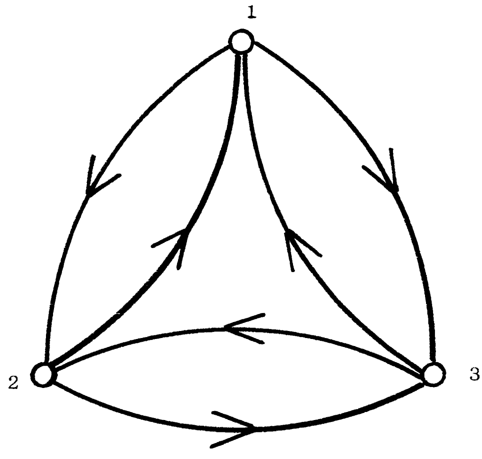

Fig. 1. A symmetric digraph.

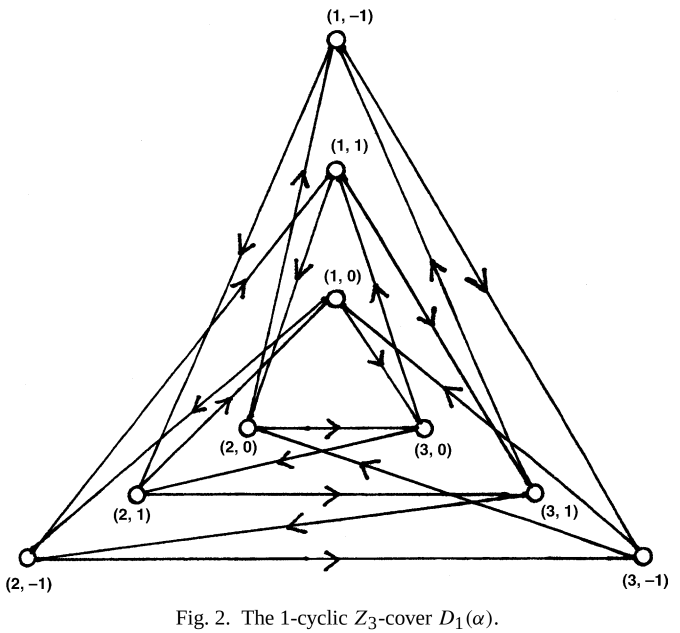

Fig. 2. The 1-cyclic Z_3-cover $D_1(\alpha)$.

The *characteristic polynomial* $\Phi(D; \lambda)$ of a digraph D is defined by $\Phi(D; \lambda) = \det(\lambda\mathbf{I} - \mathbf{A}(D))$. For a square matrix $\mathbf{B}$, we define $\Phi(\mathbf{B}; \lambda) = \det(\lambda\mathbf{I} - \mathbf{B})$. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 5. *Let D be a connected symmetric digraph, Γ a finite group, $g \in \Gamma$ and $\alpha : E(D) \longrightarrow A$ an alternating function. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the* *irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. For $h \in \Gamma$, the matrix $\mathbf{A}_h = (a_{vw}^{(h)})$ is defined as follows:*

$$a_{vw}^{(h)} = \begin{cases} 1 & \text{if } \alpha(v, w) = h \text{ and } (v, w) \in E(D), \\ 0 & \text{otherwise.} \end{cases}$$

Then the reciprocal of the zeta function of the g-cyclic Γ-cover $D_g(\alpha)$ of D is

$$\begin{aligned} \mathbf{Z}_{D_g(\alpha)}(u)^{-1} &= \det(\mathbf{I} - \mathbf{A}(D)u) \cdot \prod_{i=2}^{t} \left\{ \det\left(\mathbf{I} - u \sum_h \rho_i(h) \otimes \mathbf{A}_{hg} \right) \right\}^{f_i} \\ &= u^{n|\Gamma|} \Phi\left(D; \frac{1}{u}\right) \cdot \prod_{i=2}^{t} \left\{ \Phi\left(\sum_h \rho_i(h) \otimes \mathbf{A}_{hg}; \frac{1}{u} \right) \right\}^{f_i}. \end{aligned}$$

Proof. By Theorem 4, we have

$$\mathbf{Z}_{D_g(\alpha)}(u)^{-1} = \det(\mathbf{I} - \mathbf{A}(D_g(\alpha))u) = u^{n|\Gamma|} \Phi\left(\mathbf{A}(D_g(\alpha)); \frac{1}{u} \right).$$

By [11, Theorem 1], it follows that

$$\mathbf{Z}_{D_g(\alpha)}(u)^{-1} = u^{n|\Gamma|} \Phi\left(D; \frac{1}{u}\right) \cdot \prod_{i=2}^{t} \left\{ \Phi\left(\sum_h \rho_i(h) \otimes \mathbf{A}_{hg}; \frac{1}{u} \right) \right\}^{f_j}. \qquad \square$$

Corollary 1. $\mathbf{Z}_D(u)^{-1} \mid \mathbf{Z}_{D_g(\alpha)}(u)^{-1}$.

For a finite abelian group Γ, let Γ^* be the character group of Γ.

Corollary 2. *Let D be a connected symmetric digraph with n vertices, Γ a finite abelian group and $\alpha : E(D) \longrightarrow \Gamma$ an alternating function. Then the reciprocal of the zeta function of the g-cyclic Γ-cover $D_g(\alpha)$ of D is*

$$\mathbf{Z}_{D_g(\alpha)}(u)^{-1} = u^{n|A|} \Phi\left(D; \frac{1}{u}\right) \cdot \prod_{\chi \neq 1 \in \Gamma^*} \Phi\left(\sum_h \chi(h) \mathbf{A}_{hg}; \frac{1}{u} \right).$$

Proof. Each irreducible representation of Γ is a linear representation, and these constitute the character group Γ^*. By Theorem 5, the result follows. $\square$

For example, we consider the 1-cyclic Z_3-cover $D_1(\alpha)$ of Fig. 2. By Corollary 2, we have

$$\begin{aligned} \mathbf{Z}_{D_1(\alpha)}(u)^{-1} &= u^9(1/u^3 - 3/u - 2)(1/u^3 - 3\zeta/u - 2)(1/u^3 - 3\zeta^2/u - 2) \\ &= u^9(1/u^3 - 3/u - 2)(1/u^6 + 3/u^4 - 4/u^3 + 9/u^2 - 6/u + 4) \\ &= 1 - 6u^3 - 15u^6 - 8u^9, \end{aligned}$$

where $\zeta = (-1+\sqrt{-3})/2$. Thus

$$\begin{aligned}\log \mathbf{Z}_{D_1(\alpha)}(u) &= -\log(1-u^3(6+15u^3+8u^6))\\ &= 6u^3+33u^6+170u^9+\cdots.\end{aligned}$$

Some values of N_m are given as follows:

$$N_3 = 18,\ N_6 = 198,\ N_9 = 1530, \ldots \text{ and } N_m = 0\ (m \not\equiv 0 \bmod 3).$$

4. *L*-functions of symmetric digraphs

Let D be a connected symmetric digraph, Γ a finite group and $\alpha : E(D) \longrightarrow \Gamma$ an alternating function. Furthermore, let ρ be an irreducible representation of Γ and $g \in \Gamma$. Then we define the function $\alpha_g : E(D) \longrightarrow \Gamma$ as follows: $\alpha_g(v,w) = \alpha(v,w)g^{-1}$, $(v,w) \in E(D)$. For each path $P = (v_1, \ldots, v_l)$ of D, let $\alpha_g(P) = \alpha(v_1,v_2)g^{-1}\cdots\alpha(v_{l-1},v_l)g^{-1}$.

For $m \geqslant 1$, let $\mathscr{C}_m$ be the set of all cycles of length m in D. Set

$$N_m = \sum_{C \in \mathscr{C}_m} \operatorname{tr}(\rho(\alpha_g(C))).$$

Then, the *L-function* of D associated to ρ, α and g is defined by

$$\mathbf{Z}_D(u,\rho,\alpha,g) = \exp\left(\sum_{m \geqslant 1} \frac{N_m}{m} u^m\right).$$

Let $1 \leqslant i, j \leqslant n$. Then, the (i,j)*-block* $\mathbf{B}_{i,j}$ of an $fn \times fn$ matrix $\mathbf{B}$ is the submatrix of $\mathbf{B}$ consisting of $f(i-1)+1, \ldots, fi$ rows and $f(j-1)+1, \ldots, fj$ columns.

Theorem 6. *Let D be a connected symmetric digraph with n vertices, Γ a finite group and $\alpha : E(D) \longrightarrow \Gamma$ an alternating function. Furthermore, let ρ be an irreducible representation of Γ, and f the degree of ρ. Then the reciprocal of the L-function of D associated to ρ, α and g is*

$$\begin{aligned}\mathbf{Z}_D(u,\rho,\alpha,g)^{-1} &= \prod_{[C]} \det\left(\mathbf{I} - \rho(\alpha_g(C))u^{|C|}\right)\\ &= \det\left(\mathbf{I} - u\sum_{h \in \Gamma} \rho(h) \otimes \mathbf{A}_{hg}\right).\end{aligned}$$

Proof. At first, let

$$\eta(u) = \prod_{[C]} \det(\mathbf{I} - \rho(\alpha_g(C))u^{|C|}),$$

where $[C]$ runs over all equivalence classes of prime cycles of D. By the Jacobi formula $\det \exp \mathbf{A} = \exp \operatorname{tr} \mathbf{A}$, we have

$$\begin{aligned}
\eta(u)^{-1} &= \prod_{[C]} \det \exp \left\{ -\log(\mathbf{I} - \rho(\alpha_g(C))u^{|C|}) \right\} \\
&= \prod_{[C]} \exp \operatorname{tr} \left(\sum_{m \geqslant 1} \frac{1}{m} \rho(\alpha_g(C^m))u^{m|C|} \right) \\
&= \exp \left(\sum_{[C]} \sum_{m \geqslant 1} \frac{1}{m} \operatorname{tr}(\rho(\alpha_g(C^m)))u^{m|C|} \right) \\
&= \exp \left(\sum_{m \geqslant 1} \sum_{C} \frac{1}{m|C|} \operatorname{tr}(\rho(\alpha_g(C^m)))u^{m|C|} \right) \\
&= \exp \left(\sum_{m \geqslant 1} \frac{1}{m} N_m u^m \right) = \mathbf{Z}_D(u, \rho, \alpha, g).
\end{aligned}$$

Next, let $V(D) = \{v_1, \ldots, v_n\}$ and consider the lexicographic order on $V(D) \times V(D)$ derived from a totally order of $V(D)$: $v_1 < v_2 < \cdots < v_n$. If (v_i, v_j) is the m-th pair under the above order, then we define the $fn \times fn$ matrix $\mathbf{A}_m = ((\mathbf{A}_m)_{p,q})_{1 \leqslant p,q \leqslant n}$ as follows:

$$(\mathbf{A}_m)_{p,q} = \begin{cases} \rho(\alpha_g(v_p, v_q))u & \text{if } p = i, q = j \text{ and } (v_i, v_j) \in E(D), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathbf{B} = \mathbf{A}_1 + \cdots \mathbf{A}_k$, $k = n^2$. Then we have

$$\mathbf{B} = u \sum_h \mathbf{A}_{hg} \otimes \rho(h).$$

Let L be the set of all Lyndon words in $V(D) \times V(D)$. Then we can also consider L as the set of all Lyndon words in $\{1, \ldots, k\}$: $(v_{i_1}, v_{j_1}), \ldots, (v_{i_q}, v_{j_q})$ corresponds to $m_1 m_2, \ldots, m_q$, where (v_{i_r}, v_{j_r}) $(1 \leqslant r \leqslant q)$ is the m_rth pair. Theorem 3 implies that

$$\det(\mathbf{I}_{nf} - \mathbf{B}) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where $\mathbf{A}_l = \mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_p}$ for $l = i_1, \ldots, i_p$. Note that $\det(\mathbf{I} - \mathbf{A}_l)$ is the alternating sum of the diagonal minors of $\mathbf{A}_l$. Thus, we have

$$\det(\mathbf{I} - \mathbf{A}_l) = \begin{cases} \det(\mathbf{I} - \rho(\alpha_g(C))u^{|C|}) & \text{if } l \text{ is a prime cycle } C, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\eta(u) = \det\left(\mathbf{I}_{nf} - u\sum_{h\in\Gamma}\mathbf{A}_{hg}\otimes\rho(h)\right) = \det\left(\mathbf{I}_{nf} - u\sum_{h\in\Gamma}\rho(h)\otimes\mathbf{A}_{hg}\right).$$

Hence the result is obtained. □

By Theorems 5 and 6, the following result holds.

Corollary 3. *Let D be a connected symmetric digraph*, Γ *a finite group*, $g \in \Gamma$ *and* $\alpha : E(D) \longrightarrow \Gamma$ *an alternating function. Then we have*

$$\mathbf{Z}_{D_g(\alpha)}(u) = \prod_{\rho}\mathbf{Z}_D(u, \rho, \alpha, g)^f,$$

where ρ *runs over all irreducible representations of* Γ, *and* $f = \deg\rho$.

By Theorem 6 and Corollary 2, the following result holds.

Corollary 4. *Let D be a connected symmetric digraph*, Γ *a finite abelian group*, $g \in \Gamma$ *and* $\alpha : E(D) \longrightarrow \Gamma$ *an alternating function. Then we have*

$$\mathbf{Z}_{D_g(\alpha)}(u) = \prod_{\chi\in\Gamma^*}\mathbf{Z}_D(u, \chi, \alpha, g).$$

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